Lecture 2: Sheaves

Note Title 6/20/2019

Presheaf

Contravariant functor from $Top(X)$ to Ab — category of coherent sheaves

X: topological space

$Top(X)$ — obj: open subset of X; mor: open inclusion

$\mathcal{F}$: pre-sheaf on X

$$U \subseteq X \text{ (open)} \longmapsto \mathcal{F}(U) \text{ abelian group}$$

$$V \subseteq U \text{ (open)} \rightsquigarrow \mathcal{F}(U) \xrightarrow{\rho_{UV}} \mathcal{F}(V)$$

s.t (i) $\rho_{UU} = id_U$

(ii) $W \subseteq V \subseteq U$, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$

$\mathcal{F}$ is a sheaf if V_i open cover of X

(i) $s|_{V_i} = 0, \ \forall i \implies s = 0$

(ii) $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j} \implies \exists s$ s.t $s|_{V_i} = s_i$

Definition: ① $\mathcal{F}$ = presheaf, $p \in X$

$$\rightsquigarrow \mathcal{F}_p = \varinjlim_{p \in U} \mathcal{F}(U) \quad \text{stalk of } \mathcal{F} \text{ at } p$$

② $\mathcal{F}, \mathcal{G}$ presheaves on X

$\rightsquigarrow \varphi : \mathcal{F} \to \mathcal{G}$ morphism $\in Hom(\mathcal{F}, \mathcal{G})$

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \downarrow \rho_{UV} & & \downarrow \rho_{UV} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array} \quad \rightsquigarrow \quad \mathcal{F}_p \xrightarrow{\varphi_p} \mathcal{G}_p$$

Proposition 1: $\varphi: \mathcal{F} \to \mathcal{G}$ morphism of pre-sheaves

then φ isom. $\Leftrightarrow$ φ_p isom, $\forall p \in X$

pf: $\Rightarrow$ straight-forward

$\Leftarrow$ Goal: $\mathcal{F}(U) \xrightarrow[\simeq]{\varphi(U)} \mathcal{G}(U)$, $\forall U \subseteq X$ assuming φ_p isom.

injectivity: $\varphi(U)\, s = 0$, $s \in \mathcal{F}(U)$ $\Rightarrow$ $\varphi_p(s) = 0$, $\forall p \in U$

$\Rightarrow$ (φ_p injective) $s_p = 0$, $\forall p \in U$

$\longrightarrow$ $\exists \{V_i\}$ open cover of U, $s|_{V_i} = 0$

$\longrightarrow$ (axiom of sheaf) $s = 0$ on U

surjectivity: $s \in \mathcal{G}(U)$, $\exists$ open cover $\{V_i\}$ of U

s.t. $\exists\, t_i \in \mathcal{F}(V_i)$, $\varphi_{V_i}(t_i) = s|_{V_i}$

injectivity $\Rightarrow$ $\varphi_{V_i \cap V_j}(t_i|_{V_i \cap V_j}) = s|_{V_i \cap V_j} = \varphi_{V_i \cap V_j}(t_j|_{V_i \cap V_j})$

$\Rightarrow$ $t_i|_{V_i \cap V_j} = t_j|_{V_i \cap V_j}$ $\therefore$ $\exists t \in \mathcal{F}(U)$, $t|_{V_i} = t_i$

$\therefore$ $\varphi_U(t) = s$

Definition: $\varphi: \mathcal{F} \to \mathcal{G}$ morphism of presheaf

$\rightsquigarrow$ $\ker\varphi(U) = \ker(\varphi: \mathcal{F}(U) \to \mathcal{G}(U))$. Similarly, for $\operatorname{Im}\varphi$ & $\operatorname{Coker}\varphi$

Remark: $\operatorname{Im}\varphi$, $\operatorname{Coker}\varphi$ are presheaves but not necessarily sheaves.

ex. $\mathcal{F} = C^\infty(S^1) \xrightarrow{d} \mathcal{G} = \Omega^1(S^1)$

$f \longmapsto df$

Sheafification

To make the sheaf of abelian groups into an abelian category, one need to make "$\operatorname{Im}\varphi$, $\operatorname{Coker}\varphi$" into a sheaf

Proposition 2: $\mathcal{F}$ = presheaf

$\Rightarrow \exists! \ (\mathcal{F}^+, \theta)$ w/ universal property
sheaf $\quad \theta: \mathcal{F} \to \mathcal{F}^+$

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{G} \ \text{sheaf} \\ \theta \searrow & \circlearrowright & \nearrow \\ & \mathcal{F}^+ & \end{array}$$

pf: $\mathcal{F}^+(U) := \left\{ s: U \longrightarrow \bigcup_{p \in U} \mathcal{F}_p \ \middle| \ \begin{array}{l} \bullet\ s(p) \in \mathcal{F}_p \\ \bullet\ \exists V \underset{\text{open}}{\subseteq} U,\ t \in \mathcal{F}(V) \\ \quad \text{w/} \ s(p) = t_p,\ \forall p \in V \end{array} \right\}$

$\mathcal{F}_p \cong \mathcal{F}^+_p$

$\rightsquigarrow$ can define $\operatorname{Im}\varphi$, $\operatorname{Coker}\varphi$ $\rightarrow$ quotient sheaf

Sheaves with pointwise isomorphic stalk might NOT be isomorphic. ex: non-trivial line bundles

Definition:

- $\mathcal{F} \subseteq \mathcal{G}$ subsheaf if $\mathcal{F}(U) \subseteq \mathcal{G}(U)$, $\forall U \underset{\text{open}}{\subseteq} X$
- $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ injective if $\operatorname{Ker}\varphi = 0$
  surjective if $\operatorname{Im}\varphi = \mathcal{G}$

$\rightsquigarrow$ notion of exact sequence for sheaves

$\mathcal{F}^{i-1} \xrightarrow{\varphi_{i-1}} \mathcal{F}^{i} \xrightarrow{\varphi_i} \mathcal{F}^{i+1}$

exactness at $\mathcal{F}^i \Leftrightarrow \operatorname{Im}\varphi_{i-1} = \operatorname{Ker}\varphi_i$

$\Leftrightarrow$ exactness at $\mathcal{F}^i_p$

Proposition: $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ morphism of sheaves on X

① φ injective iff $\mathcal{F}(U) \xrightarrow{\varphi_U} \mathcal{G}(U)$ injective

② φ surjective iff $\forall U \underset{\text{open}}{\subseteq} X$, $s \in \mathcal{G}(U)$

$\exists \{U_i\}$ open cover of U

s.t. $\exists t_i \in \mathcal{F}(U_i)$. $\varphi_{U_i}(t_i) = s|_{U_i}$

almost direct from the definition of the sheafification

Definition: $f: X \to Y$ continuous map between topological spaces, $\mathcal{F}$ on X, $\mathcal{G}$ on Y

- $f_*\mathcal{F}$ on Y, $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$
- $f^{-1}\mathcal{G}$ on X, := sheaf associated to $U \mapsto \lim\limits_{V \supseteq f(U)} \mathcal{G}(V)$

$$(f^{-1}\mathcal{G})_x \cong \mathcal{G}_{f(x)}$$

Homework: chap II 1.8, 1.17, 1.18

More useful terminology

- $\operatorname{Supp}\mathcal{F} := \{p \in X \mid \mathcal{F}_p \neq 0\}$
  support of $\mathcal{F}$

- <u>Skyscrapper sheaf</u>
  $p \in X \rightsquigarrow i_p(A)$, $i_p(A)(U) = \begin{cases} A, & p \in U \\ 0, & \text{otherwise} \end{cases}$

- Sometimes denote $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$ taking sections
  $\Gamma(U, -)$ is a left-exact functor
  i.e. $0 \to \mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \implies 0 \to \Gamma(U, \mathcal{F}') \xrightarrow{\varphi_U} \Gamma(U, \mathcal{F}) \xrightarrow{\psi_U} \Gamma(U, \mathcal{F}'')$, $\forall U$

pf: 1. exactness at $\Gamma(U, \mathcal{F}')$ follows from the Proposition

2. " $\Gamma(U, \mathcal{F})$

$\boxed{\ker \psi_U \supseteq \operatorname{Im} \varphi_U}$ $\quad \psi_p \circ \varphi_p = 0 \implies \psi_U \circ \varphi_U = 0$
$\mathcal{F}''$ sheaf

$\boxed{\operatorname{Im}\varphi_U \supseteq \operatorname{Ker}\psi_U}$ $\quad s \in \ker \psi_U \subseteq \mathcal{F}(U)$

$\because \operatorname{Im}\varphi_p = \ker\psi_p$, $\exists \{U_i\}$ covering of U, $t_i \in \mathcal{F}'(U_i)$

s.t. $\varphi_{U_i}(t_i) = s|_{U_i}$

$\because \varphi_{U_i\cap U_j}(t_i|_{U_i\cap U_j}) = \varphi_{U_i\cap U_j}(t_j|_{U_i\cap U_j}) = s|_{U_i\cap U_j}$

$\varphi_{U_i\cap U_j}$ injective $\Longrightarrow$ $t_i|_{U_i\cap U_j} = t_j|_{U_i\cap U_j}$

axiom of sheaf $\Longrightarrow$ $\exists t \in \mathcal{F}'(U)$ s.t. $t|_{U_i} = t_i$

check on stalks $\varphi_U(t) = s$

This is later used to defined the right derived functor $R^i\Gamma(U,-) = H^i(U,-)$

- $f: X \longrightarrow Y$ continuous maps between topological spaces

$\mathcal{F}$ sheaf on X
$\mathcal{G}$ 〃 Y
$\Longrightarrow \operatorname{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) = \operatorname{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$

bijection of sets

f^{-1} left adjoint of f_*

f_* right adjoint of f^{-1}

pf: 1. $\exists$ natural maps $f^{-1}f_*\mathcal{F} \xrightarrow{\varphi} \mathcal{F}$, $\mathcal{G} \xrightarrow{\psi} f_*f^{-1}\mathcal{G}$ — corresponds to identity via above bijection

$f^{-1}f_*\mathcal{F}(U) = \varinjlim_{V \supseteq f(U)} f_*\mathcal{F}(V) = \varinjlim_{V \supseteq f(U)} \mathcal{F}(f^{-1}(V)) \longrightarrow \mathcal{F}(U)$, where $f^{-1}(V) \supseteq U$

then use the universal property of sheaves

both induced from restriction

$f_*f^{-1}(\mathcal{G})(U) = f^{-1}(\mathcal{G})(f^{-1}(U)) = \varinjlim_{V \supseteq f(f^{-1}(U))} \mathcal{G}(V) \longleftarrow \mathcal{G}(U)$, where $f(f^{-1}(U)) \subseteq U$

$\rightsquigarrow \mathcal{G} \to f_*f^{-1}\mathcal{G} \to (f_*f^{-1}\mathcal{G})^+$

2. $$\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$$

$$g \overset{\rho}{\longmapsto} f_* g \circ \psi$$

$$\varphi \circ f^{-1} h \overset{\rho'}{\longleftarrow\!\shortmid} h$$

$$\mathcal{G} \xrightarrow{\psi} f_* f^{-1}\mathcal{G} \xrightarrow{f_* g} f_*\mathcal{F}$$

$$f^{-1}\mathcal{G} \xrightarrow{f^{-1}h} f^{-1} f_* \mathcal{F} \xrightarrow{\varphi} \mathcal{F}$$

$$f^{-1}\mathcal{G}(U) \xrightarrow{g_U} \mathcal{F}(U), \qquad f^{-1}\mathcal{G}(U) \leftarrow \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$$

$$\mathcal{G}(V) \longrightarrow f_* f^{-1}\mathcal{G}(V) \xrightarrow{(f_* g)_V} f_*\mathcal{F}(V)$$

$$f_* f^{-1}\mathcal{G}(V) \cong f^{-1}\mathcal{G}(f^{-1}(V)) \xrightarrow{g_{f^{-1}(V)}} \mathcal{F}(f^{-1}(V)) \cong f_*\mathcal{F}(V)$$

This is used later to prove similar bijection for $\mathcal{O}_X$-module

$$X \xrightarrow{f} Y, \quad \mathcal{F} \text{ on } X, \quad \mathcal{G} \text{ on } Y$$

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$$

Example: 1. $X \xrightarrow{f} Y$ finite morphism $\quad f^* f_* \mathcal{O}_X \neq \mathcal{O}_X$

higher rank vector bundle

2. $\mathbb{P}^1_k \xrightarrow{f} \mathrm{Spec}\, k$, $\quad \mathcal{F} = \mathcal{O}(-1)$ on $\mathbb{P}^1_k$

$$f_*\mathcal{F} = H^0(\mathbb{P}^1, \mathcal{F}) = 0$$

$$f^* f_* \mathcal{F} = 0 \;\not\twoheadrightarrow\; \mathcal{F} \neq 0$$